MAT 334
Homework # 3
Due: Thursday, September 28th, 2017

Directions: Write careful solutions to each of the following problems on separate sheets of paper. (You may put more than one solution on the same sheet of paper, if you have enough room). Be sure to show all of your work. You are allowed to talk to your classmates about these problems. If you do receive help from a classmate, be sure to give them credit by noting their name on your solution. All solutions should be written in your own words, regardless of if you've received help. Partial credit is available. Each problem is worth five points.

1. Determine whether each of the following binary operations on the given set give the structure of a group. You may assume that the usual operations of addition and multiplication are associative. If it is a group, verify the axioms. If not, explain why.

(a) Define $*$ on $2\mathbb{Z}$ by $a * b = a + b$.

(b) Define $*$ on $\mathbb{Q}$ by $a * b = ab$.

(c) Define $*$ on $\mathbb{Q}^+$ by $a * b = \frac{ab}{4}$.

(d) Define $*$ on $GL_2(\mathbb{R}) = \{A | A \text{ is a } 2 \times 2 \text{ invertible matrix with real entries}\}$ by $A * B = AB$.

(e) Define $*$ on $\mathbb{C}$ by $a * b = |ab|$.

2. Suppose that G is a group with identity e. Further suppose that G has an even number of elements. Prove that there is an element $a \in G$, $a \neq e$, such that $a^2 = e$.

3. Let G be a group and let $g \in G$. Define a function i_g from G to G by

$$i_g(x) = gxg^{-1}.$$

Prove that i_g is an isomorphism of G with itself.

4. Suppose I tell you that the five-element set $\{e, a, b, c, d\}$ forms a group under some binary operation $*$. Make a table for $*$ that defines a group structure. Can you find a different group table? (That is, can you find more than one isomorphism type for a group with five elements?)