MAT 202
Assignment 3

Tuesday, February 8, 2010

For full credit on these problems, each must be submitted with a complete and clear solution, showing all of your work. You may work with other classmates on these problems, but please indicate on your assignment if you received help. Partial answers and incomplete solutions may be eligible for some partial credit, depending on the level of completeness and demonstrated understanding.

1. *Evaluate* $\int_1^3 x^2\,dx$ *using the limit definition, with* left *endpoints.*

   We will set this up using n intervals of equal width $\Delta x = \frac{3-1}{n} = \frac{2}{n}$. The integral is then

$$\begin{aligned}
\int_1^3 x^2\,dx &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \left(1 + i\left(\frac{2}{n}\right)\right)^2 \left(\frac{2}{n}\right) \\
&= \lim_{n\to\infty} \left(\frac{2}{n}\right) \sum_{i=0}^{n-1} \left(1 + 2\,i\left(\frac{2}{n}\right) + i^2\left(\frac{2}{n}\right)^2\right) \\
&= \lim_{n\to\infty} \left(\frac{2}{n}\right) \left(\sum_{i=0}^{n-1} 1 + 2\left(\frac{2}{n}\right)\sum_{i=0}^{n-1} i + \left(\frac{2}{n}\right)^2 \sum_{i=0}^{n-1} i^2\right) \\
&= \lim_{n\to\infty} \left(\frac{2}{n}\right) \left(n + 2\left(\frac{2}{n}\right)\frac{n(n-1)}{2} + \left(\frac{2}{n}\right)^2 \frac{n(n-1)(2n-1)}{6}\right) \\
&= \lim_{n\to\infty} \left(\frac{2}{n}\right) \left(\frac{6n^3}{6n^2} + \frac{12n^3 - 12n^2}{6n^2} + \frac{8n^3 - 12n^2 + 4n}{6n^2}\right) \\
&= \lim_{n\to\infty} \left(\frac{2}{n}\right) \left(\frac{26n^3 - 24n^2 + 4n}{6n^2}\right) \\
&= \lim_{n\to\infty} \left(\frac{52n^3 - 48n^2 + 8n}{6n^3}\right) \\
&= \lim_{n\to\infty} \left(\frac{26}{3} - \frac{8}{n} + \frac{4}{3n^2}\right) \\
&= \frac{26}{3}
\end{aligned}$$

2. *Use a geometric argument to evaluate each of the following integrals. A picture might help in your explanation.*

   (a) $\int_1^5 4\,dx$

   This is the area of a rectangle of width 4 and height 4, so the integral is 16.

   (b) $\int_2^6 x\,dx$

Draw a triangle in the first quadrant. The larger triangle has area $\frac{1}{2}6^2 = 18$ and the smaller triangle has area $\frac{1}{2}2^2 = 2$. So the area is the difference, which is 16.

(c) $\int_a^b 2x\,dx$, where $0 < a < b$. Answer $= b^2 - a^2$.

(d) $\int_0^a \sqrt{a^2 - x^2}\,dx$, where $a > 0$.
The expression $y = \sqrt{a^2 - x^2}$ is the upper half of the circle of radius a centered at the origin. The integral represents one-quarter of this circle, so the value of the integral is $\frac{1}{4}\,\pi\,a^2$

(e) $\int_0^3 |x - 1| + 2\,dx = \frac{17}{2}$

3. Suppose $\int_a^b f(x)\,dx = 3$ and $\int_a^b g(x)\,dx = 1$. Find

(a) $\int_a^b f(x) + g(x)\,dx$

$$\begin{aligned} \int_a^b f(x) + g(x)\,dx &= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \\ &= 3 + 1 \\ &= 4 \end{aligned}$$

(b) $\int_a^b 3f(x) + 2g(x)\,dx$

$$\begin{aligned} \int_a^b 3f(x) + 2g(x)\,dx &= \int_a^b 3f(x)\,dx + \int_a^b 2g(x)\,dx \\ &= 3\int_a^b f(x)\,dx + 2\int_a^b g(x)\,dx \\ &= 3(3) + 2(1) \\ &= 11 \end{aligned}$$