Notes on Evaluating Limits

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This is meant to be a tip sheet for evaluating limits in Calculus I as a supplement for the assignment on section 2.2.

1. Almost all limits that exist are nice, and can be done by "plugging" in the limiting value, as in problem 6 of section 2.2.

$$\begin{aligned}\lim_{x\to 3}\frac{x^2+3x-10}{3x^2+5x-7} &= \frac{3^2+3\cdot 3-10}{3\cdot 3^2+5\cdot 3-7}\\ &= \frac{9+9-10}{27+15-7}\\ &= \frac{8}{35}\end{aligned}$$

   Almost everybody got this problem correct. The only mistakes were simple computational ones. That's good!

   Now, it would not be much of a math class if everything were so easy. Occasionally, we run into a situation where if we plug in the limiting value, we do not get a nice answer. The nastiest answer we can get is the dreaded

$$\frac{0}{0}.$$

   This is an indeterminant form for the limit, which is a fancy way of saying that the limit can be ANYTHING, and so we need to do more work. Sorry, but for now, there is no way around this! (Just wait until we learn some calculus... then these problems will become EASY... and dare I say ... fun???)

2. The problems 12 - 32 even were meant to show you that by manipulating an indeterminant form of a limit, we can find out what the limit is. Here is problem 12 from Section 2.2

$$\lim_{x\to 2}\frac{x^2-4x+4}{x^2-x-2}$$

   Now if we cavalierly plug in the limiting value 2 into the expression, we see that we get 0/0. Go ahead and say your favorite four-letter word when you see this. (I'll use the word "drat") Okay, so we need to roll up our sleaves and get to work. There are a variety of techniques that we can use. For this problem, it turns out that the numerator and denominator both factor! So we have

$$\begin{aligned}\lim_{x\to 2}\frac{x^2-4x+4}{x^2-x-2} &= \lim_{x\to 2}\frac{(x-2)(x-2)}{(x-2)(x+1)} && \text{Factor top and bottom}\\ &= \lim_{x\to 2}\frac{(x-2)}{(x+1)} && \text{Cancel common factors}\end{aligned}$$

And now we can cavalierly plug in the limiting value and get

$$\lim_{x\to 2}\frac{(x-2)}{(x+1)}=\frac{2-2}{2+1}=\frac{0}{3}=0$$

Problem 20 works a little bit differently. Here we have

$$\lim_{x\to 0}\frac{\sqrt{x^2+4}-2}{x}$$

which also gives us the indeterminant form 0/0 (DRAT!) when we plug in the limiting value 0. A technique that comes up quite a bit in mathematics to make the expression *harder* than the original expression to get something nice to happen. In this case, if we multiply the top and bottom by the expression $\sqrt{x^2+4}+2$ (note the change in the sign from $-$ to $+$), then something nice does happen.

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{x^2+4}-2}{x} &= \lim_{x\to 0}\frac{\left(\sqrt{x^2+4}-2\right)\left(\sqrt{x^2+4}+2\right)}{x\left(\sqrt{x^2+4}+2\right)}\\
&= \lim_{x\to 0}\frac{\left(x^2+4\right)-4}{x\left(\sqrt{x^2+4}+2\right)}\\
&= \lim_{x\to 0}\frac{x^2}{x\left(\sqrt{x^2+4}+2\right)}
\end{aligned}$$

The nice thing about this is the troublesome factors of x in the numerator and denominator now cancel. This now gives us

$$\begin{aligned}
\lim_{x\to 0}\frac{x^2}{x\left(\sqrt{x^2+4}+2\right)} &= \lim_{x\to 0}\frac{x}{\left(\sqrt{x^2+4}+2\right)}\\
&= \frac{0}{\sqrt{0^2+4}+2}\\
&= \left(\frac{0}{4}\right)\\
&= 0
\end{aligned}$$

Problem 30 reads like

$$\lim_{x\to 0}\frac{x^2\cos(2x)}{1-\cos(x)}$$

Once again we get 0/0. dra... ah forget it. Now we have two common trig limits to play with. And it turns out we can multiply by the top and bottom by $1+\cos(x)$ and get something nice. Here we go...

$$\begin{aligned}
\lim_{x\to 0}\frac{x^2\cos(2x)}{1-\cos(x)} &= \lim_{x\to 0}\frac{x^2\cos(2x)\left(1+\cos(x)\right)}{\left(1-\cos(x)\right)\left(1+\cos(x)\right)}\\
&= \lim_{x\to 0}\frac{x^2\cos(2x)\left(1+\cos(x)\right)}{\left(1-\cos^2(x)\right)}
\end{aligned}$$

Now what the heck did that accomplish??? Well, if I rearrange our old friend the identity

$$\sin^2(x) + \cos^2(x) = 1,$$

we get

$$1 - \cos^2(x) = \sin^2(x).$$

Other than saying that you should learn to recognize such identities when they occur, there isn't much more that I can say about them. So we will substitute $\sin^2(x)$ for $1 - \cos^2(x)$ in the denominator of our expression. After that we will manipulate the expressions so that the factors of $\sin(x)$ are under the factors of x, aiming to use the $\lim_{x\to 0} \sin(x)/x = 1$ identity. So

$$\begin{aligned}
\lim_{x\to 0} \frac{x^2 \cos(2x)\,(1+\cos(x))}{(1-\cos^2(x))} &= \lim_{x\to 0} \frac{x^2 \cos(2x)\,(1+\cos(x))}{\sin^2(x)} \\
&= \lim_{x\to 0} \frac{x}{\sin(x)} \cdot \frac{x}{\sin(x)} \cdot \cos(2x)\,(1+\cos(x)) \\
&= 1 \cdot 1 \cdot 1 \cdot (1+1) \\
&= 2
\end{aligned}$$

Most of the trigonometric problems in this section amount to trying to manipulate the expression so that we get expressions of the form $\sin(x)/x$ so that we can use the identity.

All the examples above were meant to show us the the limits do exist, even when it appears that they do not. But note that in each case that the limiting value is NOT in the domain of the expression.

3. In the following problems, we are dealing with a different beast. That of piecewise defined functions. When the defining rule for the function changes, we are suspicious of the activity of the function. Problem 42 asks us to determine if a limit of a piecewise defined function exists. That is, find

$$\lim_{x\to 2} f(x), \text{ where } f(x) = \begin{cases} 3-2x & \text{if } x \le 2 \\ x^2-5 & \text{if } x > 2 \end{cases}$$

To do this, we need to evaluate the left-hand limit $\lim_{x\to 2^-} f(x)$ and the right-hand limit $\lim_{x\to 2^+} f(x)$. For the left-hand limit, we are taking values less than 2, so we use the first expression $3 - 2x$ to evaluate the limit. So

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 3 - 2x = 3 - 2(2) = -1$$

For the right-hand limit, we are using values greater than 2, so we use the second expression $x^2 - 5$. So

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2 - 5 = 2^2 - 5 = -1$$

Since both of these number exists, the limit exists and is equal to this value. That is,

$$\lim_{x \to 2} f(x) = -1$$

It is imperative that you check *both* the left and right hand limits to do this problem.

Problems 56 and 57 deal with the function

$$f(x) = \begin{cases} 2(x+1) & \text{if } x < 3 \\ 4 & \text{if } x = 3 \\ x^2 - 1 & \text{if } x > 3 \end{cases}$$

Problem 56 asks us to compute

$$\lim_{x \to 2} f(x)$$

Now, 2 is *nowhere near* the place where the function f changes its' defining rule. This happens at $x = 3$. Since 2 is (way) less than 3, we only have to consider the defining rule $2(x+1)$ to compute the limit near 2. So for this problem we have

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} 2(x+1) = 2(2+1) = 6$$

Problem 57 asks us to compute the limit at the place where f does change its' defining rule. To properly do this, we need to compute the left-hand limit and the right-hand limit. For the left-hand limit, we are using values less than 3, so we use $2(x+1)$. And so

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} 2(x+1) = 2(3+1) = 8$$

For the right-hand limit, we are using values that are greater than 3, so we use $x^2 - 1$. And so

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} x^2 - 1 = 3^2 - 1 = 8$$

Since these limits are equal, the total limit exists and equals this value. That is,

$$\lim_{x \to 3} f(x) = 8$$

Okay, my fingers are tired, and I'm sure you are tired of reading my hackneyed prose. I hope that this was somewhat helpful.