Some of my Published Papers on Magic Squares

These papers are posted here with permission from the journals

At the end of these papers, the list of all of my publication on magic squares is given.

Behforooz-Euler Knight Tour Magic Square With US Election Years

Hossein Behforooz

Utica College

Utica, New York 13502

hbehforooz@utica.edu

Permission to appear in www.mathaware.org from Journal of Recreational Mathematics
Published in the Journal of Recreational Mathematics vol. 35 (1)20-22, 2006

Year 2007 was the 300th anniversary of Leonard Euler birth year. He was a great mathematician who has contributed excellent results in mathematics, physics and other fields. At the same time, this brilliant man has spent his valuable time on recreational mathematics too. He was the first person to introduce the knight tours on the chess board and the knight tour magic squares. In the literature we can see several open or closed knight tour magic squares from him. In his birth year celebrations, we witnessed many special lectures about his life and his works. Many books and posters came out. But, I am afraid; I didn't see anything on his recreational mathematics studies. I picked up one of his closed knight tour magic square from one of Martin Gardener's book [1, p.191] and Americanized it and I fixed the following knight tour magic square with 64 US election years from 1788 to 2040. It is worth mentioning here two important points about the knight tour magic squares. From day one, people wanted to know the number of possible knight tours and knight tour magic squares. Very recently we have got the final answers to these questions. By using powerful computers, it has been shown that there are 26,534,728,821,064 distinct closed knight tours and only 140 distinct knight tour magic squares; see [3] and [4]. The second mystery was the incompleteness of these magic squares. By using the integers 1, 2, 3… 64 we have seen many open complete knight tour magic squares with magic sum 260 for all rows, columns and two diagonals. But there was no complete closed knight tour magic square with magic sum 260. In the closed case, the sum of the rows and columns are 260 but the diagonal sums are two different numbers 256 and 264. For almost 300 years it was a dream to have a complete closed knight tour magic square with magic sum 260 for all rows, columns and two diagonals. The fact is that this will not happen. The reason is so simple and obvious. On the chess board, the knight piece moves in alternate color cells (from white to black then from black to white or vice versa). So, in any closed knight tour magic square all the black cells, for example, contain odd numbers and all the white cells contain even numbers. Also, all the cells of one diagonal are white and all the cells of the other one are black. That means, one diagonal contains odd integers and the other one contain even integers. With a simple calculation, we find that 1+3+5+… + 65 =1024 and 2+4+6+ … + 64= 1056. When we divide these two totals by 4 we obtain 256 for the odd diagonal and 264 for the even diagonal and the conclusion is that we will not see a complete knight tour magic square, see also [2, p.93]. But the question is why did a great mathematician like Euler not notice this simple argument and why did people for 300 years expect to see a complete knight tour magic square with magic sum 260 for all rows, columns and two diagonals. Definitely, this problem was not similar to the Fermat Last Theorem to wait for 300 years and have a proof with more than hundred pages. This is the beauty of mathematics. Sometimes simple justifications are not visible and are left to next generations.

Remember that at the end of every research in mathematics we open many doors, roads and paths for others to start and continue that study.

References:

1. **Gardner, Martin:** Mathematical Magic Show, An MAA Spectrum Book, 1989.
2. **Pickover, Clifford:** The Zen of Magic Squares, Princeton University Press, 2002.
3. http://en.wikipedia.org/wiki/Knight's_tour
4. http://home.freeuk.net/ktn

1932	**2032**	**1956**	**2008**	**1924**	**2024**	**1948**	**1984**
1960	**2004**	**1928**	**2028**	**1952**	**1980**	**1920**	**2020**
2036	**1936**	**1996**	**1968**	**2012**	**1944**	**1988**	**1976**
2000	**1964**	**2040** *	**1940**	**1992**	**1972**	**2016**	**1916**
1788 *	**1888**	**1844**	**1864**	**1812**	**1912**	**1836**	**1872**
1848	**1860**	**1816**	**1884**	**1840**	**1868**	**1808**	**1908**
1892	**1792**	**1852**	**1824**	**1900**	**1800**	**1876**	**1832**
1856	**1820**	**1896**	**1796**	**1880**	**1828**	**1904**	**1804**

Behforooz-Euler Knight Tour Magic Square with US Election Years and magic sum 15312

Behforooz-Franklin Magic Square with US Election Years

Hossein Behforooz

Utica College

Utica, New York 13502

hbehforooz@utica.edu

Permission to appear in www.mathaware.org from Journal of Recreational Mathematics

Published in the Journal of Recreational Mathematics vol. 35 (1)37-38, 2006

Years ago, after becoming a US citizen, I got a gift (a coffee mug) with the pictures of all US presidents with their names and dates of election years around a lovely coffee mug. Immediately I thought what the heck, let's play with these numbers and create a magic square. I used 64 US election years from 1788 to 2040 and created the following 8 by 8 magic square with Benjamin Franklin style. This magic square has all Franklin Square properties. Later, just for curiosity, I colored these cells with Blue for democrats and Red for Republicans and I also put down their party initials (D for Democrats, R for Republicans, F for Federalists and W for Whigs) in each cell to make it more attractive. I noticed that the right hand half of the table is filled out with 14 D's and 18 R's (no blank cells). The left hand half has 14 D's and only 5 R's and few initials from old parties. There are 8 blank cells at the left side for the future use. In the middle of the magic square conversation which belongs to recreational mathematics (just for fun), here comes political interests and debates. I am not a fortune teller and I am not looking at this crystal ball to predict something about the future of US elections. But as we all know, most of the magic squares have symmetric properties, particularly, Benjamin Franklin squares. If our Behforooz-Franklin magic square needs to be symmetric, we should witness more R's (republicans) in the future. Who said that there are no applications for magic squares?

1992 D	**2028 ?**	**1800 D**	**1836 D**	**1864 R**	**1900 R**	**1928 R**	**1964 D**
1840 W	**1796 F**	**2032 ?**	**1988 R**	**1968 R**	**1924 R**	**1904 R**	**1860 R**
1996 D	**2024 ?**	**1804 D**	**1832 D**	**1868 R**	**1896 R**	**1932 D**	**1960 D**
1828 D	**1808 D**	**2020 ?**	**2000 R**	**1956 R**	**1936 D**	**1892 D**	**1872 R**
2004 R	**2016 ?**	**1812 D**	**1824 D**	**1876 R**	**1888 R**	**1940 D**	**1952 R**
1820 D	**1816 D**	**2012 D**	**2008 D**	**1948 D**	**1944 D**	**1884 D**	**1880 R**
1984 R	**2036 ?**	**1792 F**	**1844 D**	**1856 D**	**1908 R**	**1920 R**	**1972 R**
1848 W	**1788 F**	**2040 ?**	**1980 R**	**1976 D**	**1916 D**	**1912 D**	**1852 D**

Behforooz-Franklin magic square with US Election Years and magic sum 15312

D: Democrat R: Republican F: Federalist W: Whig

Behforooz Magic Squares
Derived from Magic-Latin-Sudoku Squares

Hossein Behforooz
Mathematics Department
Utica College
Utica, New York, 13502
hbehforooz@utica.edu

Permission to appear in www.mathaware.org from Journal of Recreational Mathematics
Published in the Journal of Recreational Mathematics vol. 36 (4)287-293, June 2012 issue.

Dedication: This article is dedicated to the mathematical games and puzzles inventor **Martin Gardner** (1914 – 2010), who was a pioneer in recreational mathematics.

Abstract:

The main motivation of this paper goes back to August 2003 when I was in Boulder, Colorado, to present a talk at the Recreational Mathematics Session of MAA Math-Fest Meeting. During one evening, I attended a magic show on Boulder's Main Street and in one part; the magician entertained the audiences with a table of numbers. I tried to learn the secret of that table and after show I asked the magician to teach me the secret of that table and, of course, he didn't tell me anything. I left the show with a 4×4 table written on my palm and tons of thoughts in my mind to discover the secret of that table ASAP. I couldn't sleep that night at all. In this paper, I will present the **same show with my own home made magic squares and I am sure that you will enjoy the show.** By using 4×4 Magic-Latin-Sudoku (MLS) squares I will produce a set of 4×4 Behforooz Magic Squares. These squares have incredible and amazing properties. These magic squares can be used to construct different type of fourth order magic squares for any given integer as a pre-assigned magic sum. Very similar to the claim that Archimedes made centuries ago " Give me a place to stand and I will move the earth" and here I am asking you to give me your wish number *S*, and I will present to you so many 4×4 magic squares with magic sum equal to your wish number *S*. Also, we can easily create curious mirror magic squares, permutation-free magic squares and upside down magic squares from these MLS squares.

Preliminaries

Magic Square: A magic square is a square matrix of numbers with the property that the sums along rows, columns, and main diagonals are all equal to *S* which is called the "magic sum".

Latin Square: A Latin square is a matrix of numbers or letters or different colors with the property that each number (letter or color) appears once and only once in each row and column. When all entries on each diagonal are distinct then it is called a double diagonal Latin square. Any numerical double diagonal Latin square is a magic square.

Sudoku Square: A Sudoku puzzle square of order *n* is an *n* × *n* grid such that each of the numbers 1, 2, …, *n* appears once and only once in each row, column and section. The most popular Sudoku squares are grids with nine square sections. If all entries on each diagonal are distinct then it is called an extreme Sudoku Square. Every extreme Sudoku square is a magic square and a double diagonal Latin square. In this case, all three squares are in one square. We call these tables Magic-Latin-Sudoku (MLS) squares.

Four by Four Magic-Latin-Sudoku Squares

We start with a simple puzzle. Complete the following Table 1 with numbers 1, 2, 3 and 4 and make it an extreme Sudoku square.

1	**2**	**3**	

Table 1

In order to make the Table 1 an extreme Sudoku puzzle square, there are only two possible cases for the second row. It must be [4, 3, 2, 1] or [3, 4, 1, 2]. Then, in each case, we have only one option for the third row and the fourth row. So, there are only two solutions to this puzzle which are the following two fourth orders Magic-Latin-Sudoku squares. To visualize the future process, I have used four different fonts or underlines for the entries of different sets {1, 2, 3, 4} and you will notice the reason and the secret of this choice later. Notice that each row, column, diagonal and section contains each number from each font or underline only once.

1	2	3	4
4	3	2	1
2	1	4	3
3	4	1	2

Table 2

1	**2**	**3**	**4**
3	**4**	**1**	**2**
4	**3**	**2**	**1**
2	**1**	**4**	**3**

Table 3

Obviously, there are 4! = 24 different cases for the first row and in each case, 2 different cases for the second row and all together there are 48 different solutions to our problem. If we consider four other digits rather than 1, 2, 3, 4 for the entries, like 2, 5, 7, 8, then there will be another 48 different solutions.

Behforooz Magic Squares

Consider one of the above solutions, say Table 2, and change the color of numbers in four sets of {1, 2, 3, 4} in four different colors.

1	2	3	4
4	3	2	1
2	1	4	3
3	4	1	2

Table 4

Then change these numbers color after color to 1, 2, 3, ..., 15, 16. The result will be the following Behforooz Magic Square (Table 5) with magic sum *S* = 34. Of course, there are many other fourth order Behforooz Magic Squares.

1	10	15	8
16	7	2	9
6	13	12	3
11	4	5	14

Table 5

Here Comes the Fun Part of the Magic Show

Suppose I ask you to give me a positive integer and you say "129". Then I immediately write down this magic square with magic sum *S* = 129.

24	**33**	**41**	**31**
42	**30**	**25**	**32**
29	**39**	**35**	**26**
34	**27**	**28**	**40**

Table 6

In this magic square, the magic sum *S*=129 appears everywhere. For example:

24+31+34+40=129, 30+25+39+35=129, 24+33+42+30=129, 33+41+30+25=129,
24+31+42+32=129, 42+32+29+26=129, 33+41+27+28=129, 24+41+29+35=129,
33+31+39+26=129, 24+42+35+28=129, 33+30+26+40=129, 33+42+26+28=129.

There are relations between the squares of the entries too:

$$\mathbf{24^2 + 33^2 + 41^2 + 31^2 + 34^2 + 27^2 + 28^2 + 40^2 = 42^2 + 30^2 + 25^2 + 32^2 + 29^2 + 39^2 + 35^2 + 26^2 = 8576,}$$

$$\mathbf{24^2 + 42^2 + 29^2 + 34^2 + 33^2 + 30^2 + 39^2 + 27^2 = 41^2 + 25^2 + 35^2 + 28^2 + 31^2 + 32^2 + 26^2 + 40^2 = 8576.}$$

If you change your number 129 to any other integer *S*, there exist other magic squares with magic sum equal to your new number *S* and the entries of these magic squares satisfy in all of the above properties.

The Secret of the Show

The secret of this mathemagic show is so simple. Choose one of those Behforooz magic squares, say Table 5. We know that for any GIVEN positive integer *S* there are unique integers *q* and *r* such that $S - 34 = 4q + r$, or $S = 34 + 4q + r$, with $r = 0, 1, 2, 3$. In Behforooz magic square (Table 5), by adding *q* to all 16 cells and *r* to four cells of {13, 14, 15, 16} we will have the following algorithm magic square with magic sum *S*

1+*q*	**10+*q***	**15+*q*+*r***	**8+*q***
16+*q*+*r*	**7+*q***	**2+*q***	**9+*q***
6+*q*	**13+*q*+*r***	**12+*q***	**3+*q***
11+*q*	**4+*q***	**5+*q***	**14+*q*+*r***

Table 7

In our example, we have 129 – 34 = 95 = 4×23 + 3. So, in Table 5, by adding 23 to all cells and 3 to four cells with entries {13, 14, 15, 16} we have obtained Table 6. Obviously, we can add 3 to other color cells {1, 2, 3, 4} or {5, 6, 7, 8} or {9, 10, 11, 12} to obtain other answers. That is why the answer is not unique. Even we can use another Behforooz magic square to present another algorithm table similar to Table 7.

Few Comments

As I mentioned it before, this subject is not brand new stuff and we can find similar procedures in the literature, see [1], [5] and [7]. But there is not any mathematical discussion in their presentations. Also, they have not mentioned the above interesting properties. The main secret in our presentation depends on the special partitioning of the entries of the Behforooz magic squares. The form of partitioning of Behforooz magic squares is important and we cannot use another type of fourth order magic squares in our algorithm. For example, we cannot use the famous Hui-Durer magic square in our procedure, because 14 and 15 are in the same row, see Behforooz [2].

Mirror Magic Squares

Now we change some of the entries of Tables 2 and 3 to other numbers and write the following two simple magic squares with same magic sum $S = 22$, (Tables 8 and 9). Then by juxtaposing them (joining them side by side together) we obtain two mirror magic squares Tables 10 and 11 with the same magic sum 242 (notice that $242=10\times22+22$). We can repeat this process and use three or more simple magic squares and obtain permutation-free magic squares with entries more than two digits. This is a note on the secret of the permutation-free magic squares in [2], [3] and [4].

9	**4**	**3**	**6**
6	**3**	**4**	**9**
2	**7**	**8**	**5**
5	**8**	**7**	**2**

Table 8

4	**3**	**7**	**8**
9	**6**	**2**	**5**
6	**9**	**5**	**2**
3	**4**	**8**	**7**

Table 9

94	**43**	**37**	**68**
69	**36**	**42**	**95**
26	**79**	**85**	**52**
53	**84**	**78**	**27**

Table 10

49	**34**	**73**	**86**
96	**63**	**24**	**59**
62	**97**	**58**	**25**
35	**48**	**87**	**72**

Table 11

The entries of these mirror magic squares satisfy the above two groups of properties. Interestingly, the diagonal entries verses non-diagonal entries of these magic squares have their own marvelous properties. For example:

$$\mathbf{94+36+85+27+68+42+79+53 = 43+37+95+52+78+84+26+69}$$

$$\mathbf{94^2+36^2+85^2+27^2+68^2+42^2+79^2+53^2 = 43^2+37^2+95^2+52^2+78^2+84^2+26^2+69^2}$$

$$\mathbf{94^3+36^3+85^3+27^3+68^3+42^3+79^3+53^3 = 43^3+37^3+95^3+52^3+78^3+84^3+26^3+69^3}$$

What a neat and beautiful stuff. Aren't they NEAT? Remember that this journal is the land of recreational mathematics and we are supposed to have FUN. Play with numbers and Enjoy! We can easily generalize these ideas to higher orders permutation-free magic squares, see Behforooz [6]. Of course there are mathematical reasons for the above three relations. The entries of the Behforooz Magic Squares are Evil and Odious integers and must satisfy in Prouhet Theorem (for more information on this, see [8]).

Upside down Magic Squares

If we use the upside down digits 0, 1, 6, 8 and 9 in our MLS squares, we can obtain interesting upside down or reversible mirror magic squares. For example, Table 12 is an example of an upside down magic square with magic sum 264.

96	**11**	**89**	**68**
88	**69**	**91**	**16**
61	**86**	**18**	**99**
19	**98**	**66**	**81**

Table 12

This mirror and upside down magic square can be obtained by juxtaposing of the following special magic-Latin-Sudoku squares with upside down integers 1, 6, 8, 9:

9	**1**	**8**	**6**
8	**6**	**9**	**1**
6	**8**	**1**	**9**
1	**9**	**6**	**8**

Table 13

6	**1**	**9**	**8**
8	**9**	**1**	**6**
1	**6**	**8**	**9**
9	**8**	**6**	**1**

Table 14

References:

[1] **Annemann, Theo.** The Book without a Name, Max Holden Publications, New York, 1931.

[2] **Behforooz, Hossein:** Permutation Free Magic Squares, J. of Recreational Mathematics, Vol. 33, No. 2, pp 103-105, 2004-2005.

[3] **Behforooz, Hossein**: Mystery in a Box, Pioneer a Magazine of Utica College, Spring 2005, pp. 18-20.

[4] **Behforooz, Hossein:** Mirror Magic Squares from Latin Squares, J. of Mathematical Gazette, Vol. 91, No. 521, July 2007, pp 316-321.

[5] **Behforooz, Hossein:** On Constructing 4 by 4 Magic Squares with Pre-assigned Magic Sum, J. of Mathematical Spectrum; Vol. 40, No. 3, 2007/2008, pp 127-134.

[6] **Behforooz, Hossein:** All Three Squares in One Square, Short Talk Sessions at the International Congress of Mathematicians, Daily Newsletter of ICM-2006, Madrid, Spain.

[7] **Benjamin, Arthur and Shermer, Michael :** Mathemagics, Lowell House Contemporary Books, Los Angeles, 1993.

[8] **Bernhard, Chris.** Evil Twins Alternate with Odious Twins, Mathematics Magazine, MAA, Vol. 82, No. 1, pp 57-62.

Behforooz-Franklin 32 by 32 Magic Square

Hossein Behforooz
Mathematics Department
Utica College, Utica, NY 13502
hbehforooz@utica.edu

Permission to appear in www.mathaware.org from Journal of Recreational Mathematics
Published in the Journal of Recreational Mathematics vol. 32 (2)107-110, 2004-2005

For many years, there were only three well known magic squares from Benjamin Franklin. The orders of them were 4 by 4, 8 by 8 and 16 by 16 (see, [1] and [2]). In a letter to his friend, Franklin called his famous 16 by 16 magic square "***The Most Magically Magical of any Magic Square EVER Made by any Mathematician***". Recently, Paul Pasles has found some other magic squares from Franklin and he screamed "Eureka, Eureka" and published them in [3] (see also, [4], [5] and [6]). For more details on Franklin magic squares and his famous letter, see [1], [2], [3], [4] and [7] and the references therein. In the literature, we witness so many articles about Franklin magic squares. There are even Ph.D. dissertations on this subject (see for example [7]). Also, there are few articles on how to make the Franklin magic squares (see [1], [2], [3], [7] and [8]). Before I came to America, I did not know about these articles and these instructions at all. But I generalized Franklin magic squares and I fixed the following 32 by 32 magic square by using 1, 2, 3, ……, 32^2=1024. The magic sum of this square is $S=32(32\times32+1)/2=16400$. This square has all the properties of the traditional Franklin magic squares mentioned in [1], [3], [4], [6] and [7]. The table is so big and cannot be fitted on one page. It is divided and printed in two halves and the 32 entries of the first row from left to right are: 817, …, 241, 272, 305, …, 720 and 753.

References:

1. W. S. Andrews, *Magic Squares and cubes*, Dover Publications, New York, 1960.
2. C. Pickover, *The Zen of Magic Squares, Circles and Stars*, Princeton University, 2002.
3. P. C. Pasles, The Lost Squares of Dr. Franklin, *American Mathematics Monthly*, 108, pp. 489-511, 2001.
4. P. C. Pasles, Franklin's Other 8-square, *Journal of Recreational Mathematics*, 31, pp. 161-166, 2002.
5. P. C. Pasles, Digging for Squares, *MAA Horizon*, April 2002.
6. P. C. Pasles, The Secret of Franklin's 8×8 "Magic" Square, *Journal of Recreational Mathematics*, 23, pp. 175-182, 1991.
7. M. M. Ahmed, How Many Squares Are There, Mr. Franklin? *American Mathematics Monthly*, 111, pp. 394-410, 2004.
8. J. Moran, *The Wonders of Magic Squares*, Vintage, New York, 1982.

Page (2)

This is the left hand side of the 32 by 32 Behforooz-Franklin Magic Square

784	**817**	**848**	**881**	**912**	**945**	**976**	**1009**	**16**	**49**	**80**	**113**	**144**	**177**	**208**	**241**
242	**207**	**178**	**143**	**114**	**79**	**50**	**15**	**1010**	**975**	**946**	**911**	**882**	**847**	**818**	**783**
782	**819**	**846**	**883**	**910**	**947**	**974**	**1011**	**14**	**51**	**78**	**115**	**142**	**179**	**206**	**243**
244	**205**	**180**	**141**	**116**	**77**	**52**	**13**	**1012**	**973**	**948**	**909**	**884**	**845**	**820**	**781**
780	**821**	**844**	**885**	**908**	**949**	**972**	**1013**	**12**	**53**	**76**	**117**	**140**	**181**	**204**	**245**
246	**203**	**182**	**139**	**118**	**75**	**54**	**11**	**1014**	**971**	**950**	**907**	**886**	**843**	**822**	**779**
778	**823**	**842**	**887**	**906**	**951**	**970**	**1015**	**10**	**55**	**74**	**119**	**138**	**183**	**202**	**247**
248	**201**	**184**	**137**	**120**	**73**	**56**	**9**	**1016**	**969**	**952**	**905**	**888**	**841**	**824**	**777**
785	**816**	**849**	**880**	**913**	**944**	**977**	**1008**	**17**	**48**	**81**	**112**	**145**	**176**	**209**	**240**
239	**210**	**175**	**146**	**111**	**82**	**47**	**18**	**1007**	**978**	**943**	**914**	**879**	**850**	**815**	**786**
787	**814**	**851**	**878**	**915**	**942**	**979**	**1006**	**19**	**46**	**83**	**110**	**147**	**174**	**211**	**238**
237	**212**	**173**	**148**	**109**	**84**	**45**	**20**	**1005**	**980**	**941**	**916**	**877**	**852**	**813**	**788**
789	**812**	**853**	**876**	**917**	**940**	**981**	**1004**	**21**	**44**	**85**	**108**	**149**	**172**	**213**	**236**
235	**214**	**171**	**150**	**107**	**86**	**43**	**22**	**1003**	**982**	**939**	**918**	**875**	**854**	**811**	**790**
791	**810**	**855**	**874**	**919**	**938**	**983**	**1002**	**23**	**42**	**87**	**106**	**151**	**170**	**215**	**234**
233	**216**	**169**	**152**	**105**	**88**	**41**	**24**	**1001**	**984**	**937**	**920**	**873**	**856**	**809**	**792**
793	**808**	**857**	**872**	**921**	**936**	**985**	**1000**	**25**	**40**	**89**	**104**	**153**	**168**	**217**	**232**
231	**218**	**167**	**154**	**103**	**90**	**39**	**26**	**999**	**986**	**935**	**922**	**871**	**858**	**807**	**794**
795	**806**	**859**	**870**	**923**	**934**	**987**	**998**	**27**	**38**	**91**	**102**	**155**	**166**	**219**	**230**
229	**220**	**165**	**156**	**101**	**92**	**37**	**28**	**997**	**988**	**933**	**924**	**869**	**860**	**805**	**796**
797	**804**	**861**	**868**	**925**	**932**	**989**	**996**	**29**	**36**	**93**	**100**	**157**	**164**	**221**	**228**
227	**222**	**163**	**158**	**99**	**94**	**35**	**30**	**995**	**990**	**931**	**926**	**867**	**862**	**803**	**798**
799	**802**	**863**	**866**	**927**	**930**	**991**	**994**	**31**	**34**	**95**	**98**	**159**	**162**	**223**	**226**
225	**224**	**161**	**160**	**97**	**96**	**33**	**32**	**993**	**992**	**929**	**928**	**865**	**864**	**801**	**800**
776	**825**	**840**	**889**	**904**	**953**	**968**	**1017**	**8**	**57**	**72**	**121**	**136**	**185**	**200**	**249**
250	**199**	**186**	**135**	**122**	**71**	**58**	**7**	**1018**	**967**	**954**	**903**	**890**	**839**	**826**	**775**
774	**827**	**838**	**891**	**902**	**955**	**966**	**1019**	**6**	**59**	**70**	**123**	**134**	**187**	**198**	**251**
252	**197**	**188**	**133**	**124**	**69**	**60**	**5**	**1020**	**965**	**956**	**901**	**892**	**837**	**828**	**773**
772	**829**	**836**	**893**	**900**	**957**	**964**	**1021**	**4**	**61**	**68**	**125**	**132**	**189**	**196**	**253**
254	**195**	**190**	**131**	**126**	**67**	**62**	**3**	**1022**	**963**	**958**	**899**	**894**	**835**	**830**	**771**
770	**831**	**834**	**895**	**898**	**959**	**962**	**1023**	**2**	**63**	**66**	**127**	**130**	**191**	**194**	**255**
256	**193**	**192**	**129**	**128**	**65**	**64**	**1**	**1024**	**961**	**960**	**897**	**896**	**833**	**832**	**769**

Page (3)

This is the right hand side of the 32 by 32 Behforooz-Franklin Magic Square

272	**305**	**336**	**369**	**400**	**433**	**464**	**497**	**528**	**561**	**592**	**625**	**656**	**689**	**720**	**753**
754	**719**	**690**	**655**	**626**	**591**	**562**	**527**	**498**	**463**	**434**	**399**	**370**	**335**	**306**	**271**
270	**307**	**334**	**371**	**398**	**435**	**462**	**499**	**526**	**563**	**590**	**627**	**654**	**691**	**718**	**755**
756	**717**	**692**	**653**	**628**	**589**	**564**	**525**	**500**	**461**	**436**	**397**	**372**	**333**	**308**	**269**
268	**309**	**332**	**373**	**396**	**437**	**460**	**501**	**524**	**565**	**588**	**629**	**652**	**693**	**716**	**757**
758	**715**	**694**	**651**	**630**	**587**	**566**	**523**	**502**	**459**	**438**	**395**	**374**	**331**	**310**	**267**
266	**311**	**330**	**375**	**394**	**439**	**458**	**503**	**522**	**567**	**586**	**631**	**650**	**695**	**714**	**759**
760	**713**	**696**	**649**	**632**	**585**	**568**	**521**	**504**	**457**	**440**	**393**	**376**	**329**	**312**	**265**
273	**304**	**337**	**368**	**401**	**432**	**465**	**496**	**529**	**560**	**593**	**624**	**657**	**688**	**721**	**752**
751	**722**	**687**	**658**	**623**	**594**	**559**	**530**	**495**	**466**	**431**	**402**	**367**	**338**	**303**	**274**
275	**302**	**339**	**366**	**403**	**430**	**467**	**494**	**531**	**558**	**595**	**622**	**659**	**686**	**723**	**750**
749	**724**	**685**	**660**	**621**	**596**	**557**	**532**	**493**	**468**	**429**	**404**	**365**	**340**	**301**	**276**
277	**300**	**341**	**364**	**405**	**428**	**469**	**492**	**533**	**556**	**597**	**620**	**661**	**684**	**725**	**748**
747	**726**	**683**	**662**	**619**	**598**	**555**	**534**	**491**	**470**	**427**	**406**	**363**	**342**	**299**	**278**
279	**298**	**343**	**362**	**407**	**426**	**471**	**490**	**535**	**554**	**599**	**618**	**663**	**682**	**727**	**746**
745	**728**	**681**	**664**	**617**	**600**	**553**	**536**	**489**	**472**	**425**	**408**	**361**	**344**	**297**	**280**
281	**296**	**345**	**360**	**409**	**424**	**473**	**488**	**537**	**552**	**601**	**616**	**665**	**680**	**729**	**744**
743	**730**	**679**	**666**	**615**	**602**	**551**	**538**	**487**	**474**	**423**	**410**	**359**	**346**	**295**	**282**
283	**294**	**347**	**358**	**411**	**422**	**475**	**486**	**539**	**550**	**603**	**614**	**667**	**678**	**731**	**742**
741	**732**	**677**	**668**	**613**	**604**	**549**	**540**	**485**	**476**	**421**	**412**	**357**	**348**	**293**	**284**
285	**292**	**349**	**356**	**413**	**420**	**477**	**484**	**541**	**548**	**605**	**612**	**669**	**676**	**733**	**740**
739	**734**	**675**	**670**	**611**	**606**	**547**	**542**	**483**	**478**	**419**	**414**	**355**	**350**	**291**	**286**
287	**290**	**351**	**354**	**415**	**418**	**479**	**482**	**543**	**546**	**607**	**610**	**671**	**674**	**735**	**738**
737	**736**	**673**	**672**	**609**	**608**	**545**	**544**	**481**	**480**	**417**	**416**	**353**	**352**	**289**	**288**
264	**313**	**328**	**377**	**392**	**441**	**456**	**505**	**520**	**569**	**584**	**633**	**648**	**697**	**712**	**761**
762	**711**	**698**	**647**	**634**	**583**	**570**	**519**	**506**	**455**	**442**	**391**	**378**	**327**	**314**	**263**
262	**315**	**326**	**379**	**390**	**443**	**454**	**507**	**518**	**571**	**582**	**635**	**646**	**699**	**710**	**763**
764	**709**	**700**	**645**	**636**	**581**	**572**	**517**	**508**	**453**	**444**	**389**	**380**	**325**	**316**	**261**
260	**317**	**324**	**381**	**388**	**445**	**452**	**509**	**516**	**573**	**580**	**637**	**644**	**701**	**708**	**765**
766	**707**	**702**	**643**	**638**	**579**	**574**	**515**	**510**	**451**	**446**	**387**	**382**	**323**	**318**	**256**
258	**319**	**322**	**383**	**386**	**447**	**450**	**511**	**514**	**575**	**578**	**639**	**642**	**703**	**706**	**767**
768	**705**	**704**	**641**	**640**	**577**	**576**	**513**	**512**	**449**	**448**	**385**	**384**	**321**	**320**	**257**

Permutation-Free Magic Squares

Hossein Behforooz
Mathematics Department
Utica College, Utica, NY 13502
hbehforooz@utica.edu
Permission to appear in www.mathaware.org from Journal of Recreational Mathematics
Published in the Journal of Recreational Mathematics vol. 33 (2)103-106, 2004-2005

Abstract: By using the Durer magic square I have made two interesting 4 by 4 magic squares such that for any permutation of the entries the results are new magic squares with the same magic sum. In this article, first I have a short comment about the history of the Durer magic square with a list of some interesting properties of this square. Finally my permutation-free magic squares will be presented.

A Brief History: There are 880 different 4 by 4 magic squares with entries 1, 2, …, 16. In 1693 Frenicle published a list of all these squares, see [1], [2], and [3]. In the literature, out of 880 magic squares, the following is the most famous one.

16	**3**	**2**	**13**
5	**10**	**11**	**8**
9	**6**	**7**	**12**
4	**15**	**14**	**1**

Yang Hui-Durer Magic Square

In Western countries, this is called the Durer magic square, because Albrecht Durer placed this magic square in his famous etching, *Melancholia*. The two central numbers in the bottom row read 1514 which is the year that Durer made the etching. See for example [2] and [4]. But in Eastern countries like China, India, or Iran, this magic square is called Yang Hui magic square which was made in China by Yang Hui in 1275, see [5]. We can find this magic square in Iranian literature too, see [6]. Like every ordinary magic square, the numbers in each row, column, and two diagonals add up to 34, which is called the magic sum. But this magic square has so many extraordinary properties. Almost any four numbers in four symmetric cells with respect to the center, add up to 34. For example:

16+13+4+1=34, 10+11+6+7=34, 3+2+15+14=34, 16+3+10+5=34, 3+5+12+14=34,
16+6+11+1=34, 3+13+6+12=34, 3+8+14+9=34, 16+3+14+1=34, 5+9+8+12=34.

I am sure that these are not new for most of the readers and all of these and some other properties can be found easily in the literature. But the following properties are very rare and are not popular. I have seen these in only two places, [7] and [8].

$$\mathbf{16 + 3 + 2 + 13 + 5 + 10 + 11 + 8 = 9 + 6 + 7 + 12 + 4 + 15 + 14 + 1,}$$

$$\mathbf{16^2 + 3^2 + 2^2 + 13^2 + 5^2 + 10^2 + 11^2 + 8^2 = 9^2 + 6^2 + 7^2 + 12^2 + 4^2 + 15^2 + 14^2 + 1^2,}$$

$$\mathbf{16^2 + 3^2 + 2^2 + 13^2 + 9^2 + 6^2 + 7^2 + 12^2 = 5^2 + 10^2 + 11^2 + 8^2 + 4^2 + 15^2 + 14^2 + 1^2.}$$

The relations between diagonal entries and non-diagonal entries are more sophisticated. From above properties, it is clear that, the sum of all diagonal numbers is equal to the sum of all non-diagonal numbers. But look at the other two lines on their squares and cubes.

$$\mathbf{16 + 10 + 7 + 1 + 13 + 11 + 6 + 4 = 3 + 2 + 8 + 12 + 14 + 15 + 9 + 5,}$$
$$\mathbf{16^2 + 10^2 + 7^2 + 1^2 + 13^2 + 11^2 + 6^2 + 4^2 = 3^2 + 2^2 + 8^2 + 12^2 + 14^2 + 15^2 + 9^2 + 5^2,}$$
$$\mathbf{16^3 + 10^3 + 7^3 + 1^3 + 13^3 + 11^3 + 6^3 + 4^3 = 3^3 + 2^3 + 8^3 + 12^3 + 14^3 + 15^3 + 9^3 + 5^3.}$$

Do you want some more? Check these out too.

$$\mathbf{2 + 8 + 9 + 15 = 3 + 5 + 12 + 14 = 34 = 2 \times 17,}$$
$$\mathbf{2^2 + 8^2 + 9^2 + 15^2 = 3^2 + 5^2 + 12^2 + 14^2 = 374 = 2 \times 11 \times 17,}$$
$$\mathbf{2^3 + 8^3 + 9^3 + 15^3 = 3^3 + 5^3 + 12^3 + 14^3 = 4624 = 2^4 \times 17^2 = 2^2\,(2+8+9+15)^2.}$$

Aren’t they beautiful? I wonder if the creator of this magic square knew these many properties. I doubt it.

<u>My Permutation-Free Magic Squares</u> The Yang Hui-Durer magic square gave me an idea to create the following magic square with three digit entries with a magic sum of 1998. I am calling this a “Permutation-Free Magic Square” because when we rearrange the digits of all entries in all cells (all in the same manner and the same order) the result will be a new magic square with the same magic sum 1998. So, by considering all permutations of the entries, we obtain six different magic squares all with one magic sum of 1998. In other words, change for example, 831 to 813 or 381 or 318 or 183 or 138, and do this rearrangement in all other fifteen cells, you will get six different magic squares with magic sum of 1998. Interestingly, all these six magic squares satisfy all of the above mentioned properties except the last two lines. This is just amazing. This time, with no doubt, I am aware of these properties. But still I wonder if there are any other properties that I do not see them and somebody will discover in the future. Who knows!

831	**326**	**267**	**574**
584	**257**	**316**	**841**
158	**683**	**742**	**415**
425	**732**	**673**	**168**

Permutation-Free Magic Square

Sometimes we prefer to eat a delicious food and enjoy it without knowing about its ingredients or recipe. I prefer not to explain how I fixed this magic square from the Durer magic square. This definitely won’t be another Fermat’s Last Theorem and interested readers can easily find the secret. Like any magic show, the trick of the work is so simple that if I explain it, I think, we lose the excitement. Every magic show has just a simple poof that the magician knows and the audiences wonder. That is it folks.

The following is another permutation-free magic square with four digit entries and its magic sum is 19998. By considering different permutations, this square produces 24 different magic squares with one common magic sum. My business is not “buy one and get one free” but a real bargain: “buy one and get 23 free”. Again here, all of these 24 squares satisfy all of the above properties except the last two lines. Very cool! This is the land of recreational mathematics and we are supposed to have FUN. Enjoy!

2247	**3514**	**8762**	**5475**
5376	**8861**	**3613**	**2148**
7851	**6386**	**1138**	**4623**
4524	**1237**	**6485**	**7752**

Permutation -Free Magic Square

Last Thought: You may ask this obvious question that, if there are any other (beside these) permutation-free magic square? The answer is “you bet there are”. We can write so many other 4 by 4 magic squares with higher digit entries which have those properties. But I could not make any 3 by 3 or 5 by 5 permutation-free magic squares with those properties.

References:

1. H. E. Dudeney: *Amusements in Mathematics*, Dover Publications, New York, pp. 119-121, 1970.
2. W. H. Benson and O. Jacoby: *New Recreations with Magic Squares*, Dover Publications, New York, p. 87 Appendix, 1976.
3. I. Peterson: *Mathematical Treks*, MAA publications, Washington DC, pp. 137-140, 2002.
4. W. S. Andrews: *Magic Squares and Cubes*, Dover Publications, New York, 1960, p. 147.
5. Yang Hui, Magic Squares in Chinese Mathematics, *Encyclopedia of the History of Science, Technology and Medicine in Non-Western Cultures*, Kluwer Academic Publishers (Editor, Helaine Selin).
6. Allameh Majlesi: *Helyatol Mottaqin* (an old Iranian book, around 1600).
7. Hashtroodi, Mohsen: *Yekan, The Iranian Mathematical Magazine*, Nowrooz Issue, 1965.
8. C. Pickover: *The Zen of Magic Squares, Circles and Stars*, Princeton University Press, 2002.

Weighted Magic Squares

Hossein Behforooz
Mathematics Department
Utica College
Utica, NY 13502
hbehforooz@utica.edu

Permission to appear in www.mathaware.org from Journal of Recreational Mathematics
Published in the Journal of Recreational Mathematics vol. 36 (4)283-286, June 2012 issue.

Dedication: This article is dedicated to the mathematical games and puzzles inventor **Martin Gardner** (1914 – 2010), who was a pioneer in recreational mathematics.

Abstract
In this paper, for the first time in the history of Magic Squares, you will be introduced the weighted magic squares, obtained by changing the numbers to weights in all cells of the magic squares and we will discuss about the centers of mass (fulcrums or pivot points) of these kinds of weighted magic squares.

New Question

In any magic square, if we hang different weights at the centers of different cells each equal to the number of cell, where would be the center of mass of these types of weighted magic squares? My first guess was that we may have two types of weighted magic squares. First type which I called them "Balanced Weighted Magic Squares" were those weighted magic squares that the centers of mass fall at the center of the magic square and the second type "Imbalanced Weighted Magic Squares" were those which their center of mass are different than the center of magic square.

Center Of Mass of Weighted Magic Squares

In calculus we have learned that for a planar lamina, the coordinates of the center of mass can be obtained by using double integrals. In a similar manner, in a discrete case with *n* distinct points $P_{i,j}(x_{i,j}, y_{i,j})$; $i, j = 1, 2\dots n$, with weights, $w_{i,j}$, the coordinates of the center of mass $G(\bar{x}, \bar{y})$ can be obtained by following double summations:

$$\bar{x} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} x_{i,j} w_{i,j}}{\sum_{i=1}^{n}\sum_{j=1}^{n} w_{i,j}}, \qquad \bar{y} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} y_{i,j} w_{i,j}}{\sum_{i=1}^{n}\sum_{j=1}^{n} w_{i,j}}. \qquad (1)$$

Now, consider a weighted magic square of order n with magic sum S. Suppose that the coordinates of the centers of cells are $P_{i,j}$ $(x_{i,j}, y_{i,j})$; i, j = 1, 3, 5,… 2n-1, with hanging weights

$w_{i,j}$; i, j = 1, 3,5…, 2n-1. Since for any natural number n, 1+ 3+ 5 +… + (2n-1) = n^2. Then the above formulas (1) will be the following formulas (like weighted arithmetic means):

$$\bar{x} = \frac{\sum xw}{\sum w} = \frac{n^2 S}{nS} = n, \quad \bar{y} = \frac{\sum yw}{\sum w} = \frac{n^2 S}{nS} = n. \qquad (2)$$

Hence $G(\bar{x}, \bar{y}) = (n, n)$ and this is the proof of the following theorem.

THEOREM: The center of the mass of every weighted magic square is the center of the magic square and there is no imbalanced weighted magic square.

The following examples illustrate the above results and calculations:

Example 1: Consider the 3×3 Lu Shu weighted magic square with magic sum 15, Table 1. In Table 2, the first two digits are the coordinates of the centers of the cells and the third digits represent the corresponding weights hanging at those centers.

4	**9**	**2**
3	**5**	**7**
8	**1**	**6**

Table 1

(1,5,4) •	**(3,5,9)** •	**(5,5,2)** •
(1,3,3) •	**(3,3,5)** •	**(5,3,7)** •
(1,1,8) •	**(3,1,1)** •	**(5,1,6)** •

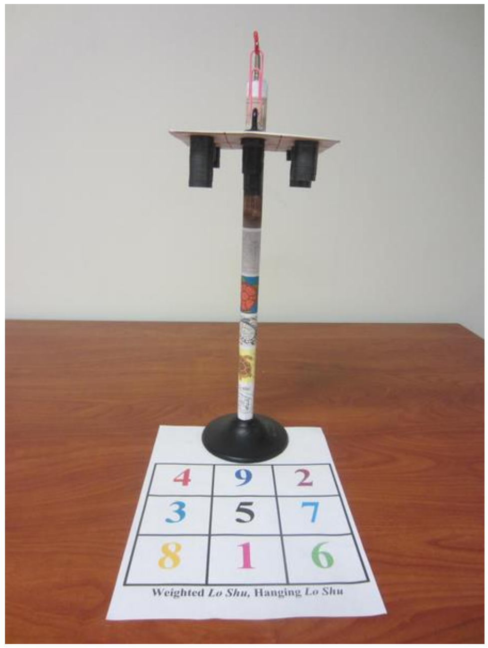
Table 2

By using the above formulas (2) for Table 2, we get:

A= (1×8+1×3+1×4) + (3×1+3×5+3×9) + (5×6+5×7+5×2) = 1×15+3×15+5×15=9×15,
B= (1×8+1×1+1×6) + (3×3+3×5+3×7) + (5×4+5×9+5×2) = 1×15+3×15+5×15=9×15,
C = (8+3+4) + (1+5+9) + (6+7+2) = 3×15. Hence

$$\bar{x} = \frac{A}{C} = \frac{\sum xw}{\sum w} = \frac{9\times15}{3\times15} = 3, \quad \bar{y} = \frac{B}{C} = \frac{\sum yw}{\sum w} = \frac{9\times15}{3\times15} = 3.$$

Therefore the coordinates of the center for mass of 3×3 Lu Shu weighted magic square is $G(\bar{x}, \bar{y})$ = (3,3) and this is true for any 3×3 weighted magic square.



Weighted Lo-Shu Hanging Lo-Shu

EXAMPLE 2: In our second example, we consider the 4×4 Behforooz magic square, Table 3, with its coordinates of the centers of cells and weights given in Table 4. Here the numbers *A*, *B*, and *C* are:

1	**10**	**15**	**8**
16	**7**	**2**	**9**
6	**13**	**12**	**3**
11	**4**	**5**	**14**

Table 3

(1,7,1) •	**(3,7,10)** •	**(5,7,15)** •	**(7,7,8)** •
(1,5,16) •	**(3,5,7)** •	**(5,5,2)** •	**(7,5,9)** •
(1,3,6) •	**(3,3,13)** •	**(5,3,12)** •	**(7,3,3)** •
(1,1,11) •	**(3,1,4)** •	**(5,1,5)** •	**(7,1,14)** •

Table 4

$$\begin{aligned} \boldsymbol{A} &= (1\times11+1\times6+1\times16+1\times1) + (3\times4+3\times13+3\times7+3\times10) + \\ &\quad + (5\times5+5\times12+5\times2+5\times15) + (7\times14+7\times3+7\times9+7\times8) = \\ &= 1\times34+3\times34+5\times34+7\times34=16\times34, \end{aligned}$$

$$\begin{aligned} \boldsymbol{B} &= (1\times11+1\times4+1\times5+1\times14) + (3\times6+3\times13+3\times12+3\times3) + \\ &\quad + (5\times16+5\times7+5\times2+5\times9) + (7\times1+7\times10+7\times15+7\times8) = \\ &=1\times34+3\times34+5\times34+7\times34 =16\times34, \end{aligned}$$

$\boldsymbol{C} = 1+2+3+\ldots+16 = 4\times34.$ Hence

$$\bar{x} = \frac{A}{C} = \frac{\sum xw}{\sum w} = \frac{16\times34}{4\times34} = 4, \quad \bar{y} = \frac{B}{C} = \frac{\sum yw}{\sum w} = \frac{16\times34}{4\times34} = 4.$$

Therefore G ($\bar{x}$, $\bar{y}$) = (4, 4). In other words, *G* ($\bar{x}$, $\bar{y}$) = (4, 4) is the center of mass for any 4×4 magic square.

Finally I am planning to invent and build the Magic Square Wind Chime Charms with their bells or pipes hanging at the center of a magic square with weights or lengths of bells or pipes are equal to the numbers of the magic squares cells. I think this will be the first practical and physically applications of the magic squares.

HOSSEIN BEHFOROOZ
List of My Publications on Magic Squares

[1] **Behforooz, Hossein**: Weighted Magic Squares, Journal of Recreational Mathematics, vol. 36 (4), 283-286, 2012.

[2] **Behforooz, Hossein**: Behforooz Magic Squares Derived from Magic-Latin- Sudoku Squares, Journal of Recreational Mathematics, vol. 36 (4), 287-293, 2012.

[3] **Behforooz Hossein:** Behforooz-Euler Magic Squares with US Election Years, Journal of Recreational Mathematics, Vol. 35, No. 1 (2009) 20-22.

[4] **Behforooz, Hossein:** Behforooz-Franklin Magic Square with US Election Years, Journal Of Recreational Mathematics, Vol. 35, No. 1 (2009) 37-38.

[5] **Behforooz, Hossein**: On Constructing 4 by 4 Magic Squares with Pre-Assigned Magic Sum, Journal of Mathematical Spectrum, Vol. 40, No. 3 (2008) 127-134.

[6] **Behforooz, Hossein**: Mirror Magic Squares from Latin Squares, Journal of Mathematical Gazette, Vol. 91, No. 521 (July 2007) 316-321.

[7] **Behforooz, Hossein**: On the Gallery of my own Magic Squares, Abstracts of the Mathematical Association of America (MAA) Seaway Section, Spring Meeting SUNY College at Oneonta, New York, (April 27-28, 2007).

[8] **Behforooz, Hossein**: All Three Squares in One: Magic Squares, Latin Squares And Sudoku Squares, Abstracts of the International Congress of Mathematicians (ICM-2006), Daily Newsletter, Madrid, Spain, August 2006.

[9] **Behforooz, Hossein**: Permutation Free Magic Squares, Journal of Recreational Mathematics, Vol. 33, No. 2 (2005) 103-106.

[10] **Behforooz, Hossein**: Behforooz-Franklin 32 by 32 Magic Square, Journal of Recreational Mathematics, Val. 33, No. 2, (2005) 107-110.

[11] **Behforooz, Hossein**: Mystery in a Box, Another Discussion on the Magic Squares, Pioneer, A Magazine of Utica College, Spring 2005.

[12] **Behforooz, Hossein**: There is Magic in Them, A Brief Introduction to Magic Squares. Inside UC Newsletter, May 2005.

[13] **Behforooz, Hossein**: Give Me any Integer M and I will give You Back a Magic Square

With Magic Sum M. Abstract of the Summer Math Fest Meeting of MAA, Providence, RI, (August 2004). Also presented at the Utica College Faculty Research Day, (Fall 2004).

[14] **Behforooz, Hossein**: Historical Notes with Interesting Properties of Some Famous Magic Squares, Abstract of the Summer Math Fest Meeting of MAA, University of Colorado at Boulder, (August 2003).

[15] **Behforooz, Hossein**: An Extension of Benjamin Franklin Magic Squares, Abstract of the International Congress of Mathematicians (ICM-2002), Beijing, China.