# Section 2.3

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5. *Identify all suspicious points and determine all points of discontinuity for*

$$f(x) = x^3 - 7x + 3$$

Since $f$ is a polynomial, there is nothing suspicious going on, and hence $f$ is continuous everywhere.

6. *Identify all suspicious points and determine all points of discontinuity for*

$$f(x) = \frac{3x+5}{2x-1}$$

When we divide, we must be careful not to divide by zero. So we should be suspicious when the denominator equals 0. That is, when

$$2x - 1 = 0.$$

When we solve for $x$, we get

$$x = \frac{1}{2}$$

So $x = \frac{1}{2}$ is a suspicious point. To determine if this is a point of discontinuity, we plug this value into the function. But we determined that $1/2$ is not in the domain of $f$. That is, $f(1/2)$ does not exist. So $x = 1/2$ is a point of discontinuity.

12. *Identify all suspicious points and determine all points of discontinuity for*

$$g(t) = \begin{cases} 3t+2 & \text{if } t \leq 1 \\ 5 & \text{if } 1 < t \leq 3 \\ 3t^2 - 1 & \text{if } t > 3 \end{cases}$$

I get suspicious when a function changes the rule for its' definition. For the piecewise defined function $g$, this occurs when $t = 1$ and $t = 3$. (At $t = 1$, the rule changes from $g(t) = 3t + 2$ to $g(t) = 5$, and at $t = 3$, the rule changes from $g(t) = 5$ to $g(t) = 3t^2 - 1$). So the suspicious points are $t = 1$ and $t = 3$.

However, these need not be points of discontinuity. Recall the definition (when in doubt, always refer to the definition) of continuity. We say a function $f$ is continuous at $x = a$ if the following three conditions hold.

1. $f(a)$ exists
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$.

We need to check these three conditions for each suspicious point $t = 1$ and $t = 3$. First we will investigate $t = 1$.

1. Since $g(1) = 3(1) + 2 = 5$, $g(1)$ indeed exists.

2. This is the part that is a little hard. To determine if the limit exists, both the left-hand limit and the right-hand limit must exist and equal the same value. So we first need to compute $\lim_{t\to 1^-} g(t)$. As we approach *from the left*, we are using the expression $3t + 2$, since numbers to the left of 1 are smaller than 1. So

$$\lim_{t\to 1^-} g(t) = \lim_{t\to 1^-} 3t + 2 = 3(1) + 2 = 5$$

We also need to compute $\lim_{t\to 1^+} g(t)$. As we approach *from the right*, we are using the expression 5, since numbers to the right of 1 are larger than 1. So

$$\lim_{t\to 1^+} g(t) = \lim_{t\to 1^+} 5 = 5$$

Since these two limits are equal, $\lim_{t\to 1} g(t)$ exists and equals 5.

3. This part is easy. Since we got 5 for both the first two parts, we have $\lim_{t\to 1} g(t) = g(1)$.

So, $g$ passed all three criteria for being continuous at $t = 1$. Hence, $g$ is continuous at $t = 1$. We must also investigate $t = 3$.

1. g(3) = 5 (so it exists)

2. We follow the same reasoning as the first part. Here we must look at the left-hand and right-hand limits. To the left of 3, we use the rule $g(t) = 5$, and to the right of 3, we use the rule $g(t) = 3t^2 - 1$. So,

$$\lim_{t\to 3^-} g(t) = \lim_{t\to 3^-} 5 = 5$$

and

$$\lim_{t\to 3^+} g(t) = \lim_{t\to 3^+} 3t^2 - 1 = 3(3)^2 - 1 = 26$$

Since these two limits are different, $\lim_{t\to 3} g(t)$ does not exist! There is no need to check the third criterion. If it fails to satisfy one, it fails.

So $g$ is not continuous at $t = 3$.

To summarize: The function $g$ has suspicious points at $t = 1$ and $t = 3$, and $g$ is discontinuous at $t = 3$.

20. *Find the value that should be assigned to $f(2)$, if any, to guarantee that $f$ will be continuous at $x = 2$, for the function*

$$f(x) = \frac{\frac{1}{x} - 1}{x - 2}$$

Note that out of the three criteria for continuity at a point $x = a$, two of them require something from $f(a)$. The other one requires something out of the limit as $x$ approaches $a$. For this problem, $a$ is 2. So let's find $\lim_{x\to 2} f(x)$. By cavalierly (not naively!) plugging in $x = 2$, we see that we get

$$f(2) = \frac{\frac{1}{2} - 1}{2 - 2} = \frac{-\frac{3}{2}}{0},$$

which is a nice reasonable number $-3/2$ divided by 0. So this limit does not exist! So there can be no value that we can assign to $f(2)$ to make this function continuous. In particular, this function has a pole at $x = 2$. That is, the graph will have a vertical asymptote at $x = 2$.

22. *Determine whether or not the given function is continuous on the prescribed interval, where*

$$f(x) = \begin{cases} x^2 & \text{if } 0 \le x < 2 \\ 3x + 1 & \text{if } 2 \le x < 5 \end{cases}$$

Here the prescribed interval is $[0, 5]$, and the only place we should be suspicious of is $x = 2$, since the function changes its definition there. To the left of 2, $f(x) = x^2$, and to the right of 2, $f(x) = 3x + 1$. To be continuous at $x = 2$, we require the three conditions to hold.

1. $f(2) = 3(2) + 1 = 7$ (which exists)
2. We check the left-hand and right-hand limits.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 = 2^2 = 4$$

and

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 3x + 1 = 3(2) + 1 = 7.$$

Since these two values are different, $\lim_{x\to 2} f(x)$ does not exist, and so the function $f$ is not continuous at $x = 2$.

32. *Show that the given equation has at least one solution on the indicated interval.*

$$\cos(x) = x^2 - 1 \text{ on } (0, \pi)$$

Let us consider the continuous function

$$f(x) = \cos(x) - x^2 + 1$$

on this interval. At the endpoints, we have

$$f(0) = \cos(0) - 0^2 + 1 = 1 - 0 + 1 = 2$$

which is positive, and

$$f(\pi) = \cos(\pi) - (\pi)^2 + 1 = -1 - \pi^2 + 1 = -\pi^2$$

which is negative. So by the Intermediate Value Theorem, there exists some $c \in (0, \pi)$ such that $f(c) = 0$. Or in other words, There is a $c \in (0, \pi)$ such that

$$\cos(c) - c^2 + 1 = 0,$$

or

$$\cos(c) = c^2 - 1.$$